# How the odd terms in the Fibonacci sequence stack up

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*Dedicated to H. N. V. Temperley on the occasion of his ninetieth birthday*
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The authors of the recent *Note* [8] exhibit an *odd* preference. They derive recurrence relations for the odd terms, $u_n = F_{2n+1}, n \geq 0$, in the sequence of Fibonacci numbers, $F_n$, defined by

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0; \quad F_0 = 0, \quad F_1 = 1.$$

In particular, they show that

$$u_{n+2} = 3u_{n+1} - u_n, \quad n \geq 0; \quad u_0 = 1, \quad u_1 = 2. \tag{1}$$

(Our notation here is not quite the same as that in [8]: in addition to introducing $F_0$, our $u_n$ has a subscript one less than its counterpart in [8].) But much the same holds for the even terms; and what can be done for the bisections of sequences given by second order linear recurrence relations with constant coefficients can be extended to other divisions as well.

However, the sequence of odd terms, $u_n$, does have some special claim to attention in its own right in the context of structures of natural combinatorial interest *seemingly* not simply defined in terms of those objects that are well-known to give rise to the Fibonacci numbers (some attempt at a thorough, systematic exploration of one such setting has been given recently in [2], and more recently still in the book-length treatment [3]). While this may not be so obvious or familiar, it does at least permit combinatorial interpretations of recurrence relations such as (1) of a more concrete kind that are not without interest, practically and, for that matter, perhaps pedagogically (this is the purpose in [2, 3], which do not, however, venture into quite the same territory we consider here). Amusingly enough, when we came upon [8], we were ourselves examining just one of these structures, comparatively simple arrangements of square cells known as *stacks*. We suspect that other readers are also likely to find this approach diverting, so record some of our observations in order to provide the stimulus of fresh challenges — we have aimed, in writing this account, and even more in illustrating it, to build a serviceable bridge from the classroom into some current research of relevance to computer science and bioinformatics.

To be precise, a stack is a cellular structure consisting of horizontal layers of contiguous cells of the square grid, each layer containing no more cells than the layer below it, except for a foundational layer that may contain an arbitrary number of cells. As such, a stack is an example of a *polyomino*, the term, derived by analogy with domino, for any finite union of cells in the square grid having simply connected interior. Some small stacks are illustrated in Figure 1. Of course, the name draws on the resemblance with a neatly balanced pile of dishes where none is larger than the one on which it rests or has more than one dish resting on it. In fact, stacks seem to have been introduced some fifty years ago, pre-dating the term *polyomino*, by Neville Temperley, as a two-dimensional model in the study of roughness of crystal surfaces, the crystal stack being thought of as growing by the accretion of cells, without overhangs or cavities (see [13]).

In this early work on crystals, stacks were viewed number theoretically as a special type of partition, and so grouped according to their number of cells. However, the ensuing enumerative results are somewhat unwieldy. But a little judicious experimentation reveals that, in contrast, classification of stacks by *perimeter* works rather neatly. As it happens, we encounter below another equinumerous class of polyominoes where the appropriate index is the number of cells. But then the bijection that we present between the two classes makes perimeter the corresponding choice for stacks.

Thus, we notice, from Figure 1, that the numbers of stacks with perimeters 4, 6, 8 and 10 are 1, 2, 5, and 13, respectively. This suggests that we are off to a promising start as regards the odd terms in the Fibonacci sequence. Indeed, we claim that the number of stacks with perimeter $2(n+2)$ is $u_n$. It is exactly here that a recurrence relation like (1) acquires practical import. For, if we are right in our claim, (1) prompts thought of some form of *triplication* of the stacks as their perimeter is increased by two, followed by the elimination of those repeated in this enlargement process; at least, that is how the equation seems to speak to us in *combinatorial* idiom.

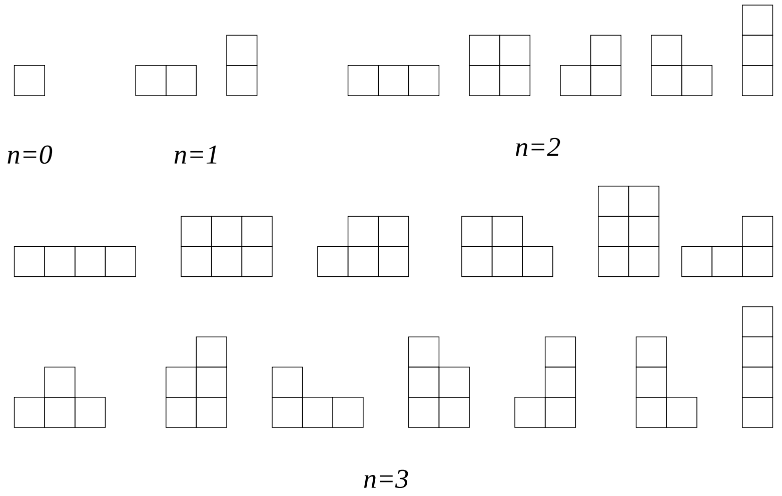


Figure 1: Small stacks

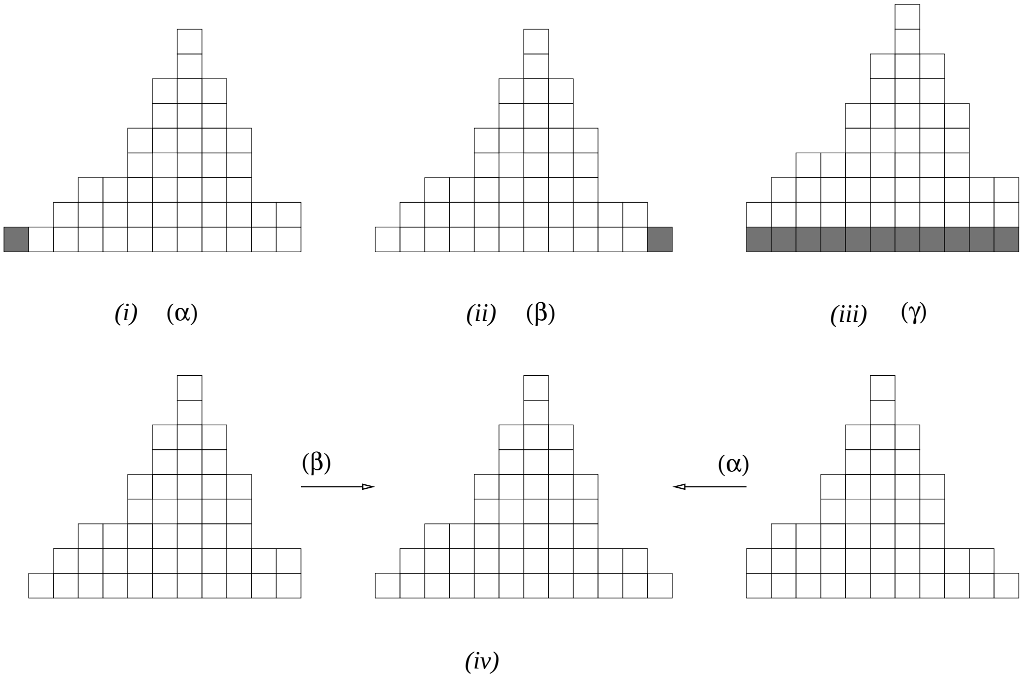


Figure 2: Growth of stacks

Now, in any stack, if the foundational layer does not stick out a bit to either side, it must have the same number of cells as the layer above it. This *trichotomy*, that the foundational layer of a stack sticks, if not to the left or right, then to the bottom, means that, starting from a single cell, we can certainly create *every* stack by successive application in some order of the following three enlargement procedures
**($\alpha$):** adding a new cell to the foundational layer on the left;
**($\beta$):** adding a new cell to the foundational layer on the right; or
**($\gamma$):** inserting a new foundational layer under the former foundational layer and with the same number of cells.
Moreover, applying each procedure separately to any given stack produces three new stacks in which the perimeter is increased by two, so that this triplication meshes with classification by perimeter.

These procedures, which are clearly reversible, are sketched in Figure 2(i)–(iii). Unfortunately our trichotomy is inexact, since, as we see in Figure 2(iv), there is some ambiguity as to the originating stack precisely when the foundational layer of the new stacks sticks out on both sides. But, in this case, excising a cell at both ends of the foundational layer, we can have any of the stacks in which the perimeter is two less than where we started (so, in all, four less than where we end up). Combining these observations, we see that triplication of the stacks of given perimeter by means of our three procedures produces all the stacks of perimeter two larger, but with some duplicates equal in number to the stacks of perimeter two less. Thus, we have captured (1), complete with initial conditions in view of Figure 1, and our claim is proved (stacks with given perimeter appear in Figure 1 in the order they are generated from those with perimeter two less according to our listing of the enlargement procedures for stacks, with repetitions suppressed as they arise).

If it is preferred to avoid this ambiguity that is the price of having an inexact trichotomy, we can obtain a *partition* of the set of stacks with perimeter $2(n+4)$ by operating instead with a simple logical alternative: either the foundational layer projects to the left from under the layer above it, or it does not. In the former case, by excising the left-most cell from the foundational layer we decrease the perimeter by two and any of the $u_{n+1}$ stacks with perimeter $2(n+3)$ is possible at this stage. In the latter case, the foundational layer projects to the right from under the layer above by $r$ cells, for some $r \geq 0$, and then removal of the entire foundational layer decreases the perimeter by $2(r+1)$, with any of the $u_{n+1-r}$ stacks with perimeter $2(n+3-r)$ possible at this stage. Since these cases are exclusive and exhaustive, we obtain the *renewal recurrence relation*:

$$\begin{aligned} u_{n+2} &= u_{n+1} + \sum_{r=0}^{n+1} u_{n+1-r} \\ &= 2u_{n+1} + u_n + u_{n-1} + \cdots + u_0, \quad n \geq 0. \end{aligned} \tag{2}$$

By differencing (2), we recover (1) (equation (2) does not feature in [8], but for this style of argument, compare [9]):

$$\begin{array}{rl} u_{n+2} & = 2u_{n+1} + u_n \quad + u_{n-1} + \cdots + u_0 \\ u_{n+1} & = \qquad\quad + 2u_n + u_{n-1} + \cdots + u_0 \\ \hline u_{n+2} - u_{n+1} & = 2u_{n-1} - u_n \end{array}$$

Now, an attractive aspect of a renewal recurrence relation like (2) is that it can be derived by a process of peeling back stage by stage to *discover* what objects might still be missing at any stage. Thus, in order to avoid duplications at the outset in starting with the stacks with perimeter $2(n+3)$, we might only allow, say, procedures $(\alpha)$ and $(\gamma)$. This gives $2u_{n+1}$ distinct stacks with perimeter $2(n+2)$, accounting for the first term on the right of (2). We have now taken care of extending the foundational layer to the left and lifting it without any extension on either side. So, in addition, we must take into consideration lifting the foundational layer while also extending it on the right. As the perimeter of the lifted stack diminishes, this extension to the right must lengthen in order to produce a stack with perimeter $2(n+4)$. A schematic representation in terms of what might be thought of as successive *generations* of stacks is given in Figure 3, each new generation bringing in immediate descendants of the first generation to account for the further final term in (2).

By tracing the *genealogy* of stacks in this way, we discover that, for each $r$, with $1 \leq r \leq n+1$, we also need to insert under each of the $u_{n+1-r}$ stacks with perimeter $2(n+4-r)$ a new foundational layer which is flush on the left with the layer above it but juts out by $r$ cells on the right, as illustrated, in a typical case, in Figure 4. Hence, we recover successively the remaining terms on the right of (2), in conformity with our earlier, direct line of reasoning.

But, whichever line of argument we take in establishing (2), the argument moves from a partition of a set of stacks to a dissection of the stacks themselves, as though,

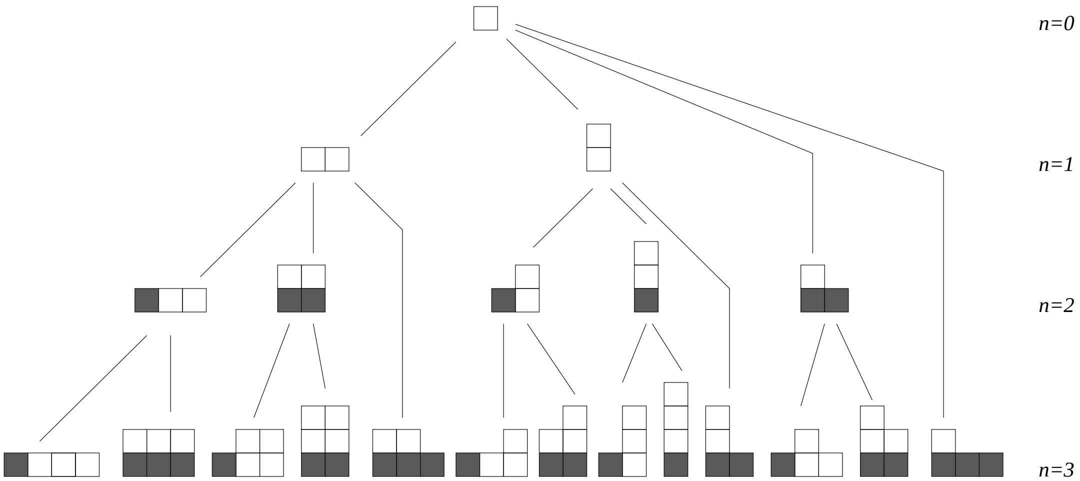


Figure 3: Generation of stacks — a renewal approach

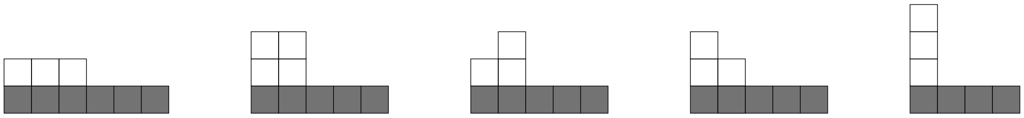
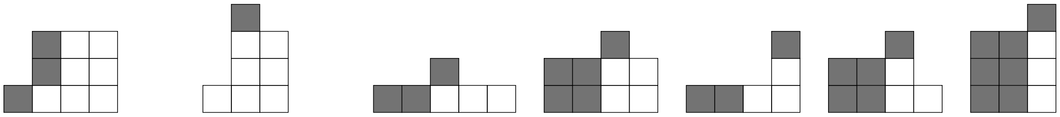


Figure 4: A renewal approach: $n = 4$, $r = 2$

indeed, we really were shelling them into single cells jutting out to the left and long strips underneath projecting to the right. In fact, renewal recurrence relations like (2) are typically associated with some kind of *factorization* of objects into sub-objects that are not further decomposable in terms of this factorization. The dissections of the stacks resulting from our demonstrations of (2) can be viewed in this light, even if it may appear less natural when counting according to perimeter rather than to the number of cells (a more natural illustrative example in this sense is the problem in tiling described in [9]).

Our second, recursive line of argument leading to (2) comes more into its own when it may not be so apparent how the partition proceeds, or even if a partition will work out in this way at all. The reader might care to consider, by way of an exercise on this point, deriving (2) by deploying other enlargement procedures for stacks. For example, as alternatives to the combination of ($\alpha$) and ($\gamma$), we might either extend *each* layer of a stack by one cell on the left, as in Figure 5(i), or add a single cell above the left-most cell in the top layer, as in Figure 5(ii). Starting from the stacks with perimeter $2(n+3)$, these new procedures again give $2u_{n+1}$ stacks with perimeter $2(n + 4)$. As before, this initiates the recursive production of successive generations of stacks indexed by perimeter. It is then a matter for the reader to check through genealogies, after manner displayed in Figure 3, that, for each $r$, with $1 \leq r \leq n+1$, we also need to extend each layer in a stack with perimeter $2(n + 4 - r)$ by $r$ cells on the left, while adding a single cell above the left-most cell in the top layer (for a typical example, see Figure 5(iii), although there is no substitute to actually trying to create the counterpart to Figure 3).

A further curious observation concerns layers (rows) consisting of a single cell, for



*(i)* *(ii)* *(iii)* the contribution $u_2$ when $n$=3, $r$=1

Figure 5: A second renewal approach

which it is convenient to introduce the term *vestigial layer.* At this juncture, our interest in them is piqued on observing that the number, $u_n^*$, say, of these vestigial layers among all $u_n$ stacks with perimeter $2(n+2)$ is again the $n$-th odd Fibonacci number, that is

$$u_n^* = u_n, \quad n \geq 0, \tag{3}$$

as can be verified by inspection, at least in the first few cases, from Figure 1. But we were drawn to (3) by an analogous result in an application in theoretical computer science described in [1, §3] to which we shall come more fully later, in the context of another class of cellular structures.

Armed with (1), as demonstrated in terms of our three procedures $\alpha$, $\beta$, and $\gamma$, this supplementary enumeration falls out easily. For, applying procedure ($\alpha$) or ($\beta$) to the stacks with perimeter $2(n+3)$ preserves all $u_{n+1}^*$ vestigial layers, except in the case of the *column* stack all layers of which vestigial, since then the foundation layer contains two cells. On the other hand, applying procedure ($\gamma$) to these stacks again preserves all $u_{n+1}^*$ vestigial layers but adds a further vestigial layer of a single cell under the column stack. For the combined application of procedures ($\alpha$) and ($\beta$), the effect is to diminish the total number of vestigial layers by one. Hence, on the lines of our combinatorial demonstration of (1), taking these various contributions together, we have

$$u_{n+2}^* = 2(u_{n+1}^* - 1) + (u_{n+1}^* + 1) - (u_n^* - 1), \quad n \geq 0. \tag{4}$$

This means that $u_n^*$ satisfies the *same* recurrence relation as $u_n$, as it is apparent, from (4), that the adjustments just described cancel out. The two enumerations must therefore yield the *same* sequence, since their initial conditions are also identical, thus confirming (3). Of course, for those who delight in bijections, the real issue here is, not these enumerations *per se*, but rather to match up vestigial layers among all stacks with perimeter $2(n+2)$ with these stacks themselves.

This last comment is apposite, in that the stacks of perimeter $2(n+2)$ are by no means the only combinatorial objects to have attracted attention that turn out to be enumerated by the odd terms in the Fibonacci sequence. Some of these objects are admittedly rather recondite, although that has not proved a deterrent to a recent attempt [6] to give a comprehensive collection of bijections between them (an endeavour that appears all the more remarkable in that what is, in effect, the defining recurrence relation (1) is not mentioned explicitly). But, even in the study of cellular structures, this sequences already arises in what is perhaps one of the simplest of the *recursively defined* varieties, what, for want of a more succinct and visual word, we dub *fronds* (compare Figure 6 and, more suggestively, Figure 7).

A *frond* is then a cellular structure (polyomino) in the square grid that grows from a single square cell by placing additional square cells one at a time either (i) at the left end of the top layer, to extend that layer, or (ii) above any cell in the top layer, thereby starting a new top layer. Fronds have been studied previously with rows and columns interchanged under the technical name of *directed column-convex polyominoes*, as, for example, in [1] (which, its title notwithstanding, also never notes (1) explicitly) and, more recently, in [6]; but interest in cellular structures of this kind can also be traced back to pioneering work [14] of Neville Temperley some fifty years ago. The first few small fronds are shown in Figure 6. The *area* of a frond is the number of cells it contains. As is borne out by Figure 6 in the early cases, the key result is that the number of fronds of area $n+1$ is $u_n, n \geq 0$.

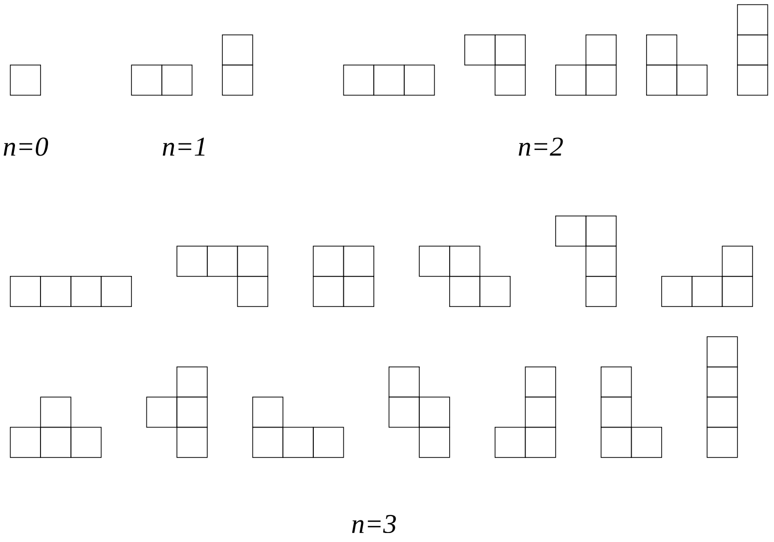
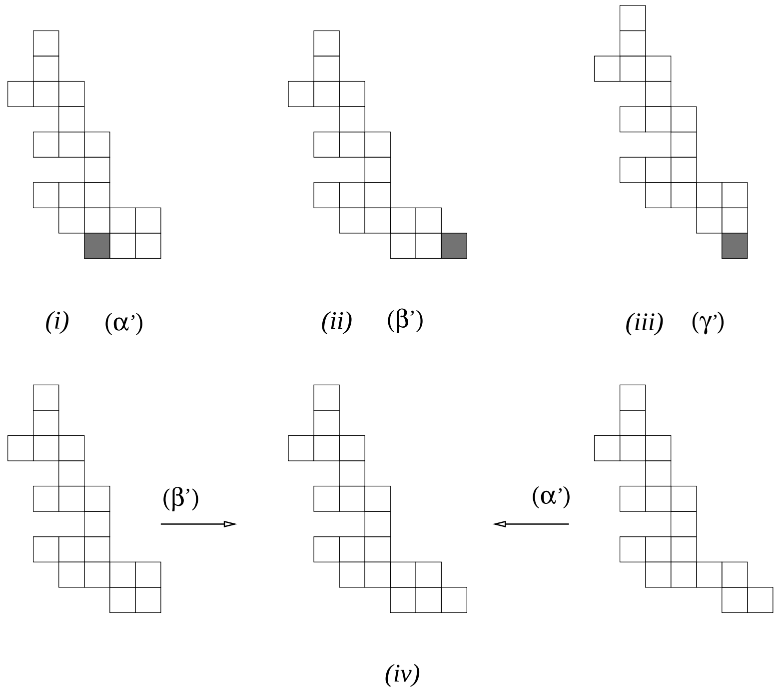


Figure 6: Small fronds

Once again (and in contrast with [1, 6]), we find (1) suggestive of an approach to the enumeration of fronds by area through some form of *triplication*. Indeed, it is very natural to mimic the procedures ($\alpha$), ($\beta$), and ($\gamma$) for triplicating stacks. Given a frond, we obtain three new fronds in which the area is increased by one as follows
**($\alpha'$):** adding a new cell to the foundational layer on the left (note that this new cell can be tucked under the next layer of the frond and need not project out to the left of it);
**($\beta'$):** adding a new cell to the foundational layer on the right (note that this new cell will project out to the right from the next layer of the frond); or
**($\gamma'$):** inserting a new cell under the cell at the extreme right of the foundational layer, thereby creating a new foundational level. As before in the case of stacks, it is the interplay of the first two of these procedures that leads to ambiguity. So, imitating the argument already presented for stacks, the upshot of this way of looking at the generation of fronds is to establish (1), thus confirming the assertion concluding our previous paragraph — a visual depiction is offered in Figure 7 which is to be compared, in this regard, with Figure 2 in the comparable discussion of stacks (again, fronds with given area in Figure 6 appear in the order they are generated from those with area one less according to our listing of the enlargement procedures for fronds).



*(i)* (α') *(ii)* (β') *(iii)* (γ')

*(iv)*

Figure 7: Growth of fronds

There is less point in going through the suggested demonstration of (1) for fronds because, as might be guessed, that stacks and fronds admit such similar *triplications* helps unlock a bijection between the stacks with perimeter $2(n+2)$ and the fronds with area $n+1$, both sets being enumerated, as we have now determined, by the odd terms in the Fibonacci sequence, $u_n = F_{2n+1},\ n \geq 0$. Clearly, the first pair of enlargement procedures for stacks is, in effect, the same as the first pair of enlargement procedures for fronds; these procedures govern the number of cells in a layer to the left and right of the layer above it. Similarly, the third procedure in either case controls the depth of layers from the top layer. In essence, then, to obtain a bijection, we need only to squeeze out the contribution that procedure $(\gamma)$ makes to the number of cells in a layer so as to convert it to procedure $(\gamma^{'})$.

Thus, for each layer in a stack, we first produce a *compressed* layer. For a start, the top layer is its own compressed layer (by convention, we leave the cell on the extreme right unlabelled and label the others, if any, $\alpha$; compare the example displayed in Figure 8). Then, for each subsequent layer going down, the compressed layer consists of the cells to the left (labelled $\alpha$) and right (labelled $\beta$) by which it extends beyond the layer above it, separated by a single distinguished cell (labelled $\gamma$). A frond is then obtained from these compressed layers by placing the distinguished cell of one compressed layer under the cell at the extreme right of the previous compressed layer, starting from the top (as illustrated in Figure 8; also compare [6, Fig. 1], with columns re-oriented as rows). This frond is thus a compressed expression of the stack in terms of the enlargements procedures: if $n$ instances of our procedures are required to obtain the stack starting from an initial cell, the associated frond will have $n+1$ cells — the unlabelled cell on the extreme right of the top layer and $n$ labelled cells — while the stack itself will have perimeter $2n+4 = 2(n+2)$, since each procedure increases the perimeter by two.

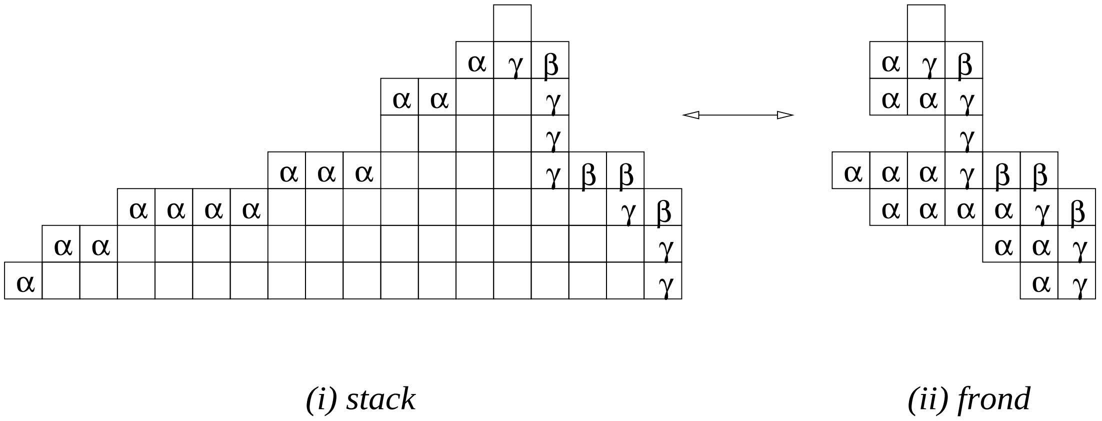


Figure 8: Bijection between stacks and fronds

Indeed, this frond of compressed layers captures in skeletal outline the stack from which it is defined, as can be made more visible through the *profile* of a stack. The *right profile* of a stack is the path of cells in it from the cell at the extreme left of its top layer to that at the extreme right of its foundational layer, where the rule is to move across layers to their far right, before moving down to the next layer; the *left profile* is defined analogously. It is then conspicuous that the associated frond recapitulates the right profile of its defining stack, with, in effect, the stack's left profile squashed on to it. Thus, in recovering the stack from its frond, we already have the right profile. Then, in any layer of the frond, the number of cells to the left of this right profile gives the extent to which that layer in the stack juts out to the left from under the layer above it. In fact, the fronds in Figures 6 and 7 stand in just this bijective relation to the stacks in Figures 1 and 2.

In a stack, a *vestigial* layer (consisting of a single cell) is part of the right profile of the stack, so is preserved in the associated frond. However, as can be seen in Figures 6, 7, and 8, fronds can contain other vestigial layers; this happens whenever the layer in the corresponding stack has the same number of cells as the layer above it. The problem of determining the total number, $h_n$, say, of vestigial layers among fronds of area $n+1$ is considered in [1, §3]. The application in mind there is the random generation of fronds, and $h_n$ is used in ascertaining the average number of calls to a routine employed in generating fronds (although, of course, this is to translate the problem in [1, §3] into our present terminology). Now, for us, finding $h_n$ becomes an exercise based on the enlargement procedures and the demonstration of (1) based on them, after the manner of our account of (4).

Suppose we start from the $u_{n+1}$ fronds with area $n+2$. Procedures $(\alpha')$ and $(\beta')$ preserve vestigial layers, except in the $u_n$ cases where the foundational layer itself contains only a single cell. On the other hand, procedure $(\gamma')$ not only preserves existing vestigial layers, but adds a single new cell as a foundational layer, that is, adds a new vestigial layer. Hence, on the lines of (4), we have, for $n \geq 1$,

$$h_{n+2} = 2(h_{n+1} - u_n) + (h_{n+1} + u_{n+1}) - (h_n - u_{n-1}).$$

But, in view of (1) itself, this simplifies to

$$h_{n+2} = 3h_{n+1} - h_n + u_n, \tag{5}$$

which holds now for $n \geq 0$. Finally, eliminating $u_n$ by means of (1), leads to the recurrence relation

$$h_{n+4} - 6h_{n+3} + 11h_{n+2} - 6h_{n+1} + h_n = 0, \quad n \geq 0, \tag{6}$$

subject to the initial conditions

$$h_0 = 1, \quad h_1 = 2, \quad h_2 = 6, \quad h_3 = 18.$$

Although $h_n$ is tabulated in [1, p. 292], neither (5) nor (6) appear in that paper. As a further exercise in very much the same spirit, we note that, in comparison with (5), it can be shown that the total number of layers, $t_n$, alike for stacks with perimeter $2(n+2)$ or fronds of area $n+1$ satisfies

$$t_{n+2} = 3t_{n+1} - t_n + u_{n+1}, \quad n \geq 0.$$

It follows that $t_n$ also satisfies (6), but with different initial conditions:

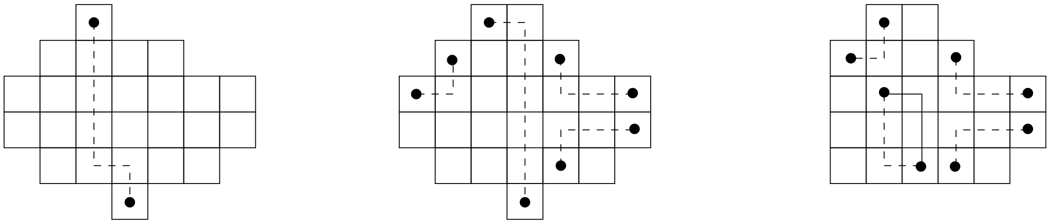
$$t_0 = 1, \quad t_2 = 3, \quad t_3 = 10, \quad t_3 = 32.$$

In drawing to a close, it is perhaps in order to set our observations in some context, in the form of some end-notes that provide supplementary information for the interested reader who might like to explore matters further.

First of all, when we encountered [8], we were investigating another class of cellular structures, the so-called *L-convex polyominoes*, recently introduced by Castiglione and Restivo in [4]. These are the polyominoes in which any two cells can be connected by a path of cells in the polyomino that switches direction between the vertical and the horizontal at most once — such paths with one change of direction look like the letter $L$ in one of its four cyclic orientations, hence the name. It is clear that the rows and columns of an $L$-convex polyomino are contiguous strips of cells of the square grid. Polyominoes whose rows and columns have this latter property are called *convex.* These definitions are made more apparent in Figures 9(i) and 9(ii). Moreover, as illustrated in Figure 9(iii), where we need never go across and down, we may not actually need to use all four orientations of the $L$. Indeed, the stacks, as introduced here in other terms, can now be seen to be exactly those $L$-convex polyominoes for which the we can get by with just two orientations, the $L$ itself (down and across) and its vertical mirror image (across and up), for those paths where a change of direction is required. Now, as found in [5], the number $w_n$ of $L$-convex polyominoes with perimeter $2(n+2)$ turns out to satisfy a recurrence relation rather like (1):

$$w_{n+2} = 4w_{n+1} - 2w_n, \quad n \geq 1; \quad w_1 = 2, \quad w_3 = 7.$$

So, what of the subset, intermediate between the stacks and the full set of $L$-convex polyominoes, for which some three specified orientations of the $L$ suffice to handle all the paths between cells which involve a change of direction? Here our curiosity was further quickened on noticing the re-emergence of what seems to be $h_n$ in looking at the



*(i) not L−convex* *(ii) L−convex* *(iii) three orientations suffice*

Figure 9: Convex polyominoes

polyominoes with perimeter $2(n+2)$ in this intermediate class (as subsequently confirmed in [10]).

Secondly, taking up the discussion in [11, p. 214] and [7], it has become almost *de rigueur* in any enumerative work of this kind to cite the *On-Line Encyclopedia of Integer Sequences* [12], where the sequence $u_n = F_{2n+1}$ of odd terms in the Fibonacci sequence is $A001519$, while the Fibonacci sequence is $A000045$. For that matter, $t_n$ and $v_n$ also have their own entries, $A038731$ and $A003480$, while yet another sequence sharing the same recurrence relation (6), but with first four terms 1, 5, 19, 65, appears as $A001870$. But, at the time of first writing, the sequence $h_n$ had not acquired an entry (it has since entered the ranks as $A094864$). Similarly, $w_n$ is also present, as $A003480$, but without allusion to the enumeration of polyominoes. The sequence arising from the enumeration of stacks by number of cells inaugurated in [13] is $A001523$.

Thirdly, for the authors of [8], (1) was a stepping stone to other equations involving the odd terms, $u_n$, in the Fibonacci sequence. So, it is natural to wonder whether a combinatorial account (proof) can be given of those other equations, too, perhaps in terms of the stacks we have used in explicating (1) (and (2))? (Such a thorough-going commitment to the combinatorial interpretation of equations is very much the agenda of [2, 3], to which clearly a recurrence relation like (6) provides a greater challenge than, say, (1) or (2).)

Finally, in opening our second paragraph, we suggested that objects, like stacks and fronds, which, as we have now seen give rise to the odd terms of the Fibonacci sequence, are structures of natural combinatorial interest *seemingly* not simply defined in terms of those objects that are well-known to give rise to the Fibonacci numbers. Still, it might be that the *full* Fibonacci sequence is lurking somewhere close after all. Readers must judge for themselves what they find *natural* or not. Certainly, the even terms of the Fibonacci sequences also appear in several guises in the enumeration of stacks: for example, the number of stacks with perimeter $2(n+2)$ having at least two layers with the foundational layer flush on the left with the layer above it is easily seen to be $F_{2n}$ for $n \geq 1$. But that hardly seems especially *natural* in itself, any more than the further observation that

$$h_n = t_{n-1} + F_{2n}, \quad n \geq 1.$$

Perhaps we come closer to something like a natural appearance of *all* the terms of the Fibonacci sequence from the perspective of the top layer of the stacks and fronds. For, in the ordering of the stacks in Figure 1 or the fronds in Figure 5, among the polyominoes with perimeter $2(n+2)$, those in which the top layer contains more than one cell come together, before those with vestigial top layer; and there are $F_{2n-1}$ of the former and $F_{2n}$ of the latter.

# References

[1] E. Barcucci, R. Pinzani, and R. Sprugnoli, Directed column-convex polyominoes by recurrence relations, *in* M.-C. Gaudel and J.-P. Jouannaud, eds., *TAPSOFT '93: Theory and Practise of Software Development. Proceedings of Fourth International Joint CAAP/FASE Conference, Orsay, France,* $13^{th}-17^{th}$ *April, 1993. Lecture Notes in Computer Science, Vol. 668* (Springer, Berlin, 1993), pp. 282–298.

[2] A. T. Benjamin and J. J. Quinn, The Fibonacci numbers — exposed more discretely, *Math. Mag.*, **76** (2003), 182–192.

[3] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof. Dolciani Mathematical Expositions Vol. 27* (Math. Assoc. Amer., Washington, DC, 2003).

[4] G. Castiglione and A. Restivo, Reconstruction of $L$-convex polyominoes, *Electronic Notes in Discrete Math.*, **12** (2003).

[5] G. Castiglione, A. Frosini, A. Restivo, and S. Rinaldi, Enumeration of $L$-convex polyominoes, (preprint).

[6] E. Deutsch and H. Prodinger, A bijection between directed column-convex polyominoes and ordered trees of height trees of height at most three, *Theoretical Computer Science*, **307** (2003), 319–325.

[7] E. K. Lloyd, Letter regarding [11], *Math. Gaz.*, **83** (1999), 134.

[8] V. Ragesh and G. Leversha, Note 88.02: Some properties of the odd terms of the Fibonacci sequence, *Math. Gaz.*, **88** (2004), 85–86.

[9] D. G. Rogers, Things fall apart: a recurrence of tiling, *The Mathematics Teacher*, (to appear).

[10] D. G. Rogers, Three ells in a polllyomino, (in preparation).

[11] J. Sharp, Have you seen this number?, *Math. Gaz.*, **82** (1998), 203–214.

[12] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at ⟨http://www.research.att.com/∼njas/sequences/⟩.

[13] H. N. V. Temperley, Statistical mechanics and the partition of numbers II: The form of crystal surfaces, *Proc. Cambridge Philos. Soc.*, **48** (1952), 683–697.

[14] H. N. V. Temperley, Combinatorial problems suggested by the statistical mechanics of domains and of rubber-like molecules, *Phys. Rev. (2)*, **103** (1956) 1-16.